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On the duality principle in pseudo-Riemannian Osserman manifolds

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Abstract

Here we give a natural extension of the duality principle for the curvature tensor of pointwise pseudo-Riemannian Osserman manifolds. We proved that this extended duality principle holds under certain additional assumptions. Also, it is proved that duality principle holds for every four-dimensional Osserman manifold.
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1. Introduction

Let (M, g) be an n-dimensional pseudo-Riemannian manifold of the signature $(s, n-s)$. Let $\varepsilon_X = g(X, X)$ be the norm of the vector $X \in T_pM$. Depending on their norm, we distinguish the following types of tangent vectors: spacelike ($\varepsilon_X > 0$), timelike ($\varepsilon_X < 0$), null ($\varepsilon_X = 0$, $X \neq 0$), definite ($\varepsilon_X \neq 0$) and unit ($|\varepsilon_X| = 1$). By S_pM, S_p^+M, and S_p^-M we will denote all unit non-null, spacelike and timelike vectors in T_pM, respectively.

Let ∇ be the Levi-Civita connection and let R be the associated Riemannian curvature tensor; $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Jacobi operator $\mathcal{R}_X : Y \longrightarrow R(Y, X)X$ is a symmetric endomorphism of the tangent bundle TM. For non-null X, $\mathcal{R}_X$ preserves the orthogonal space $\{X\}^\perp$, and we will use the notation $\mathcal{R}'_X$ for the restriction of $\mathcal{R}_X$ to this space.

One says that (M, g) is *timelike* (respectively, *spacelike*) *pointwise Osserman* if the characteristic polynomial of $\mathcal{R}_X$ is independent of $X \in S_p^-M$ (respectively, S_p^+M). The notions pointwise timelike Osserman and pointwise spacelike Osserman are equivalent and, if (M, g) is either of them, then (M, g) is said to be *Osserman*. Manifolds such that the characteristic polynomial of $\mathcal{R}_X$ is constant on the bundle S^-M (respectively, S^+M) of unit timelike (respectively, spacelike) vectors are called *globally timelike* (respectively, *spacelike*) *Osserman*. In the higher signature setting, unlike in the Riemannian setting, the eigenvalue structure does not determine the conjugacy class of a symmetric operator but its Jordan normal form. We say that (M, g) is *timelike pointwise Jordan Osserman*

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(respectively, *spacelike pointwise Jordan Osserman*) if the Jordan normal form of $\mathcal{R}_X$ is independent of $X \in S_p^+M$ (respectively, on S_p^-M). Similarly, we define *globally timelike Jordan Osserman manifolds*. While the notions of globally timelike Osserman or globally spacelike Osserman are the same, the notions globally timelike Jordan Osserman and globally spacelike Jordan Osserman are distinct. For more details about this topic, see [9,10].

In Riemannian settings, in a series of papers by Chi and by Nikolayevsky it was shown that globally Osserman manifolds of dimension $n \neq 16$ are two-point homogeneous spaces; and consequently for $n \neq 16$, this gives an affirmative answer to *Osserman conjecture* (see [17,18,7,13–15]). For a generalization of Osserman conjecture to the pseudo-Riemannian case, see [4].

Let R be an *algebraic curvature operator*. This is a tensor satisfying the curvature symmetries

$$R(X, Y) + R(Y, X) = 0, \tag{1}$$

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \tag{2}$$

$$g(R(X, Y)Z, W) = g(R(Z, W)X, Y). \tag{3}$$

One says R is an *Osserman algebraic curvature tensor* if the associated Jacobi operator has the characteristic polynomial constant on the unit pseudo-spheres S_p^-M and S_p^+M. Similarly, R is a *timelike (spacelike) Jordan Osserman algebraic curvature tensor* if the associated Jacobi operator has the Jordan normal form constant on the unit pseudo-sphere S_p^-M (S_p^+M). One of the most natural approaches to studying the Osserman-type problems was suggested in [12]. It has two steps:

(1) classifying the Osserman (Jordan Osserman) algebraic curvature tensors;
(2) finding those Osserman (Jordan Osserman) algebraic curvature tensors which can be realized as the curvature tensors of a pseudo-Riemannian manifold.

In the Riemannian settings, the *duality principle* is the following property of an Osserman algebraic curvature tensor R:

$\lambda \in \mathbb{R}$ satisfies the duality principle if, for any unit vectors X and Y, there holds

$$\mathcal{R}_X Y = \lambda Y \quad \textit{if and only if} \quad \mathcal{R}_Y X = \lambda X.$$

We say that R satisfies the duality principle if and only if every eigenvalue of Jacobi operator $\mathcal{R}_X$, $X \in \mathbb{S}^{n-1}$, satisfies the duality principle.

The duality principle, which is proved by the second author (see [20]) and re-proved by Gilkey (see [10]), is used as one of the important tools in the proof of Osserman conjecture for $n \neq 8, 16$ by Nikolayevsky (see [14]).

The main result of this paper is a natural extension of the notion of the duality principle on pseudo-Riemannian manifolds.

It is worth noting that the Osserman manifolds are described in Riemannian settings (except some cases for $n = 16$) and in Lorentzian settings (spaces of constant sectional curvature; see [2]), but in the case of higher signature (except the case of four-dimensional Kleinian manifolds; see [3,9]) we are very far from the complete picture. For example, the Osserman conjecture does not hold; see [19,6,3,9]. The interesting questions which arise from our investigations are: classification of all algebraic curvature tensors which satisfy the duality principle and classification of all pseudo-Riemannian manifolds whose curvature tensor satisfies the duality principle.

Our paper is organized as follows. Section 1 is devoted to the introduction in this topic, motivation, as well as some interesting questions which arise from our investigations. Since k-stein manifolds are closely related to the Osserman manifolds, in Section 2 we give some basic characterizations of k-stein conditions and especially of 1-stein and 2-stein manifolds. It is known that spacelike and timelike Jordan Osserman algebraic curvature tensors are necessarily diagonalizable in signature (p, q), $p \neq q$ (see [11]), and therefore the duality principle becomes of great interest when the Jacobi operator $\mathcal{R}_X$ is diagonalizable for any $X \in S_pM$. We call such manifolds diagonalizable pseudo-Riemannian Osserman manifolds. Following the proof of the duality principle in the Riemannian case (see [20]), we obtain the analogous conditions (Theorem 2.2) for diagonalizable pseudo-Riemannian Osserman manifolds. In Section 3 we give the definition of the duality principle, and extend it for all non-null vectors (Theorem 3.2). A characterization of the duality principle for diagonalizable pseudo-Riemannian manifolds is given in Theorem 3.3. As consequences of this, we obtain that the duality principle holds in diagonalizable pseudo-Riemannian manifolds: (i1) with all distinct eigenvalues of the Jacobi operator (Corollary 3.4) and (i2) when for any $X \in S_pM$ the Jacobi operator $\mathcal{R}_X$ has not null eigenvectors (Corollary 3.5). In Section 4 we prove that the duality principle holds for any four-dimensional Osserman manifold.

2. Preliminaries

2.1. k-stein

We say that a manifold M is *k-stein* if there exist constants C_t for $1 \le t \le k$ such that, for all $X \in SM$, $\mathrm{Tr}(\mathcal{R}_X^t) = (\varepsilon_X)^t C_t$ holds. It is well known that 1-stein manifolds are Einstein, and vice versa, and also that the Osserman timelike (spacelike) condition is equivalent to the k-stein condition (see, for example, [10, 1.7.3 Lemma]).

Let $(E_1, E_2, \ldots, E_n)$ be an arbitrary pseudo-orthonormal basis of T_pM. Let $X = \alpha E_i + \beta E_j$, where i and j are fixed and $1 \le i \ne j \le n$. Then we have

$$\mathcal{R}_X = \alpha^2 \mathcal{R}_i + \alpha\beta \mathcal{R}_{ij} + \beta^2 \mathcal{R}_j \tag{4}$$

where we put

$$\mathcal{R}_i = \mathcal{R}_{E_i} \quad \text{and} \quad \mathcal{R}_{ij} = R(\cdot, E_i)E_j + R(\cdot, E_j)E_i.$$

Now, after the substitution $\alpha^2 = \varepsilon_X \varepsilon_i - \beta^2 \varepsilon_i \varepsilon_j$ in (4), we obtain

$$\mathcal{R}_X = \varepsilon_X \varepsilon_i \mathcal{R}_i + \alpha\beta \mathcal{R}_{ij} + \beta^2 (\mathcal{R}_j - \varepsilon_i \varepsilon_j \mathcal{R}_i). \tag{5}$$

Let us introduce the following notations:

$$A^i_{pq} := [\mathcal{R}_i]_{pq} = \varepsilon_p R_{qiip}, \qquad B^j_{pq} := [\mathcal{R}_j]_{pq} = \varepsilon_p R_{qjjp}, \qquad Z^{ij}_{pq} := [\mathcal{R}_{ij}]_{pq} = \varepsilon_p (R_{qijp} + R_{qjip}). \tag{6}$$

Since

$$[\mathcal{R}_X^k]_{pq} = \sum_{p_2, p_3, \ldots, p_k} [\mathcal{R}_X]_{pp_2} [\mathcal{R}_X]_{p_2 p_3} \cdots [\mathcal{R}_X]_{p_k q},$$
$$\mathrm{Tr}(\mathcal{R}_X^k) = \sum_{p_1, p_2, \ldots, p_k} [\mathcal{R}_X]_{p_1 p_2} [\mathcal{R}_X]_{p_2 p_3} \cdots [\mathcal{R}_X]_{p_k p_1},$$

Eq. (5) puts the k-stein condition in the following form:

$$\varepsilon_X^k C_k = \sum_{p_1, \ldots, p_k, p_{k+1} = p_1} \prod_{1 \le t \le k} \left(\varepsilon_X \varepsilon_i A^i_{p_t p_{t+1}} + \alpha\beta Z^{ij}_{p_t p_{t+1}} + \beta^2 (B^j_{p_t p_{t+1}} - \varepsilon_i \varepsilon_j A^i_{p_t p_{t+1}}) \right). \tag{7}$$

Lemma 2.1. (i1) *A manifold M is 1-stein iff, for all $1 \le i \ne j \le n$, the following formulas hold*

$$\sum_{1 \le p \le n} Z^{ij}_{pp} = 0 \quad \text{and} \quad \sum_{1 \le p \le n} (B^j_{pp} - \varepsilon_i \varepsilon_j A^i_{pp}) = 0. \tag{8}$$

(i2) *If a manifold is 2-stein then, for all $1 \le i \ne j \le n$, the following formulas hold:*

$$\sum_{1 \le p,q \le n} A^i_{pq} Z^{ij}_{qp} = \sum_{1 \le p,q \le n} B^j_{pq} Z^{ij}_{qp} = 0, \tag{9}$$

$$2 \sum_{1 \le p,q \le n} A^i_{pq} B^j_{qp} - 2\varepsilon_i \varepsilon_j \sum_{1 \le p,q \le n} A^i_{pq} A^i_{qp} + \sum_{1 \le p,q \le n} Z^{ij}_{pq} Z^{ij}_{qp} = 0, \tag{10}$$

$$\sum_{1 \le p,q \le n} A^i_{pq} A^i_{qp} = \sum_{1 \le p,q \le n} B^j_{pq} B^j_{qp}. \tag{11}$$

Proof. The proof for both statements follows from (7) by specialization with $k = 1$ and $k = 2$, respectively. In the first case we just consider the terms with $\alpha\,\beta$ and β^2; in the second case we manipulate using the formulas with the terms $\alpha\,\beta$, β^2, $\alpha\,\beta^3$, and β^4. ◇

2.2. Diagonalizable manifolds

Let us consider the case when the Jacobi operator $\mathcal{R}_X$ is diagonalizable for any $X \in S_pM$. We will call such manifolds *diagonalizable Osserman* (pseudo-Riemannian) manifolds. Since spacelike and timelike Jordan Osserman algebraic curvature tensors are necessarily diagonalizable in signature (p, q), $p \neq q$, the duality principle becomes especially interesting in the diagonalizable setting.

It is clear that all information about R are encoded in the terms with $\alpha^i \beta^{i+2t}$ ($i = 0, 1$ and $t \in \mathbb{N}$) of $\mathrm{Tr}(\mathcal{R}_X^k)$ (see (7)); one can consider terms with $\alpha\beta$ and β^2. We obtain the following very important theorem, which is a generalization of the main theorem of [20].

Theorem 2.2. *Let (M, g) be a diagonalizable pseudo-Riemannian Osserman manifold, and let $(E_1, E_2, \ldots, E_n)$ be an orthonormal basis of T_pM such that $\mathcal{R}_1$ has a diagonal matrix with respect to this basis, and let $\Lambda_a = \{p \mid [\mathcal{R}_1]_{pp} = a\}$. Then, for every eigenvalue a of $\mathcal{R}_1$, and for all $1 \leq i \neq j \leq n$, there hold*

$$\text{(i1)} \sum_{p\in\Lambda_a} Z^{ij}_{pp} = 0, \tag{12}$$

$$\text{(i2)} \sum_{p,q\in\Lambda_a} Z^{ij}_{pq} Z^{ij}_{qp} = 0. \tag{13}$$

Proof. Since the proof of the main theorem of [20] does not depend on a particular form of elements Z^{ij}_{pq}, we conclude that this proof also works in the pseudo-Riemannian case. ◇

Remark 1. Let us remark that the above theorem is a reformulation of the main theorem (Theorem 1.1) of [20], or more precisely that the duality principle in the Riemannian case follows from (12) and (13).

3. Duality

Definition 3.1. Let R be an Osserman algebraic curvature tensor. For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if, for all mutually orthogonal unit vectors X, Y, there holds

$$\mathcal{R}_X(Y) = \varepsilon_X \lambda Y \Longrightarrow \mathcal{R}_Y(X) = \varepsilon_Y \lambda X. \tag{14}$$

If the duality principle holds for all $\lambda \in \mathbb{R}$, then we say that the duality principle holds for the algebraic curvature tensor R (or for the pseudo-Riemannian Osserman manifold (M, g) whose curvature tensor is R).

Remark 2. It is obvious that the above definition is a natural generalization of the duality principle for Riemannian settings; see [20].

In the next theorem we will show that one can extend the domain for X and Y in the case of diagonalizable Osserman manifolds.

Theorem 3.2. *Let (M, g) be a diagonalizable Osserman manifold such that the duality principle holds for $\lambda \in \mathbb{R}$. Then implication (14) holds for all $X, Y \in T_pM$ with $\varepsilon_X \neq 0$.*

Proof. The proof has three steps. Each of them is an extension of the implication (14) to all

(i1) $X, Y \in T_p(M)$ with $\varepsilon_X, \varepsilon_Y \neq 0$ and $g(X, Y) = 0$;
(i2) $X, Y \in T_p(M)$ with $\varepsilon_X \neq 0$ and $g(X, Y) = 0$;
(i3) $X, Y \in T_p(M)$ with $\varepsilon_X \neq 0$.

For (i1) it is enough to rescale any two non-null orthogonal vectors X and Y which satisfy (14).

(i2) Let Y be a null eigenvector ($\varepsilon_Y = 0$) of $\mathcal{R}_X$ for the eigenvalue $\varepsilon_X\lambda$ orthogonal to X. Since $\mathcal{R}_X$ is diagonalizable, by the general theory of symmetric endomorphisms in pseudo-unitarian spaces (see, for example, [16, Section 28.1]), there exists a mixed, orthogonal to X, eigenplane for the eigenvalue $\varepsilon_X\lambda$ which contains Y. This implies that there are non-null mutually orthogonal vectors $E, F \in \{X\}^{\perp}$ such that $\mathcal{R}_X(E) = \varepsilon_X \lambda E$, $\mathcal{R}_X(F) = \varepsilon_X \lambda F$ and $Y = E + F$. Now we have

$$\mathcal{R}_{E+\theta F}(X) = \mathcal{R}_E(X) + \theta^2 \mathcal{R}_F(X) + \theta\,(R(X,E)F + R(X,F)E)\,. \tag{15}$$

Since for $\theta^2 \neq 1$ we have that $E + \theta F$, E and F are three non-null eigenvectors of $\mathcal{R}_X$ which correspond to the eigenvalue $\varepsilon_X \lambda$, then for them the duality principle hold, which implies that $\mathcal{R}_{E+\theta F}(X) = \varepsilon_{E+\theta F}\lambda X$, $\mathcal{R}_E(X) = \varepsilon_E \lambda X$ and $\mathcal{R}_F(X) = \varepsilon_F \lambda X$. From (15) we have

$$\varepsilon_{E+\theta F}\lambda X = \varepsilon_E \lambda X + \theta^2 \varepsilon_F \lambda X + \theta\,(R(X,E)F + R(X,F)E)$$

and, since $\varepsilon_{E+\theta F} = \varepsilon_E + \theta^2 \varepsilon_F$, the equation above gives

$$\theta(R(X,E)F + R(X,F)E) = 0.$$

Consequently, for all θ, Eq. (15) becomes

$$\mathcal{R}_{E+\theta F}(X) = \mathcal{R}_E(X) + \theta^2 \mathcal{R}_F(X),$$

and especially for $\theta = 1$ we have

$$\mathcal{R}_{E+F}(X) = \mathcal{R}_E(X) + \mathcal{R}_F(X).$$

Since $\mathcal{R}_E(X) + \mathcal{R}_F(X) = \varepsilon_E \lambda X + \varepsilon_F \lambda X = \varepsilon_{E+F}\lambda X$ and $Y = E + F$, we have finally

$$\mathcal{R}_Y(X) = 0 = \varepsilon_Y \lambda X,$$

and this completes the proof of (i2).

(i3) Let us assume that $\mathcal{R}_X(Y) = \varepsilon_X \lambda Y$ for $\varepsilon_X \neq 0$ and $g(X,Y) \neq 0$. Then there exist $Z \in \{X\}^{\perp}$ and $\alpha \neq 0$ such that $Y = \alpha X + Z$. Since Y is eigenvector of $\mathcal{R}_X$, we have,

$$\mathcal{R}_X(Z) = \mathcal{R}_X(\alpha X + Z) = \varepsilon_X \lambda(\alpha X + Z),$$

and because of $\mathrm{Im}(\mathcal{R}_X) \subseteq \{X\}^{\perp}$ we have

$$0 = g(\mathcal{R}_X(Z), X) = g(\varepsilon_X \lambda(\alpha X + Z), X) = g(\varepsilon_X \lambda \alpha X, X) = \varepsilon_X^2 \alpha \lambda.$$

Now $\alpha, \varepsilon_X \neq 0$ implies $\lambda = 0$, which means that $\mathcal{R}_X(Z) = 0$ for $Z \perp X$ and $\varepsilon_X \neq 0$, but then duality (i2) implies that $\mathcal{R}_Z(X) = 0$. Now, we have

$$\begin{aligned}\mathcal{R}_Y(X) &= \mathcal{R}_{\alpha X+Z}(X) = R(X, \alpha X + Z)(\alpha X + Z) = R(X,Z)(\alpha X + Z)\\ &= -\alpha R(Z,X)X + R(X,Z)Z = -\alpha \mathcal{R}_X(Z) + \mathcal{R}_Z(X) = 0,\end{aligned}$$

which completes the proof of the theorem. $\diamond$

Let (M,g) be a diagonalizable pseudo-Riemannian Osserman manifold and let $(E_1, E_2, \ldots, E_n)$ be a pseudo-orthonormal basis of T_pM such that the Jacobi operator $\mathcal{R}_1$ has a diagonal matrix with respect to this basis. Then the duality applied to the coordinate eigenvectors gives

$$\mathcal{R}_1(E_j) = \varepsilon_1 \lambda E_j \Longrightarrow \mathcal{R}_j(E_1) = \varepsilon_j \lambda E_1. \tag{16}$$

$\mathcal{R}_1(E_j) = \varepsilon_1 \lambda E_j$ is equivalent to $\mu_j = A^1_{jj} = \varepsilon_1 \lambda$ and $A^1_{kj} = 0$ for $k \neq j$. From (6) it follows that $\lambda = \varepsilon_1 \varepsilon_j R_{j11j}$ and $R_{j11k} = 0$ for $k \neq j$. On the other hand, we have that $\mathcal{R}_j(E_1) = \varepsilon_j \lambda E_1$ is equivalent to $B^j_{11} = \varepsilon_j \lambda$ and $B^j_{k1} = 0$ for $k \neq 1$, then (6) gives $\lambda = \varepsilon_j \varepsilon_1 R_{1jj1}$ and $R_{1jjk} = 0$ for $k \neq 1$. Since $\varepsilon_1 \varepsilon_j R_{j11j} = \varepsilon_j \varepsilon_1 R_{1jj1}$ hold, we see that

$$\left(R_{j11k} = 0 \text{ for } k \neq j\right) \Longrightarrow \left(R_{1jjk} = 0 \text{ for } k \neq 1\right), \tag{17}$$

is the sufficient condition for (16). Now, we can formulate the following theorem.

Theorem 3.3. *Let (M,g) be a diagonalizable pseudo-Riemannian Osserman manifold, and let $(E_1, E_2, \ldots, E_n)$ be an orthonormal basis of T_pM such that $\mathcal{R}_1$ has a diagonal matrix with respect to this basis. The duality principle holds for (M,g) iff $Z^{1j}_{pp} = 0$ holds for all $j > 1$ and all $1 \leq p \leq n$.*

Proof. Let us consider the condition $Z_{pp}^{1j} = 0$. By (6), this means that

$$0 = Z_{pp}^{1j} = \varepsilon_p(R_{p1jp} + R_{pj1p}) = 2\varepsilon_p R_{1ppj},$$

and therefore,

$$Z_{pp}^{1j} = 0 \Longleftrightarrow R_{1ppj} = 0. \tag{18}$$

Let us assume that the duality principle holds for (M, g). Since all E_j are eigenvectors of $\mathcal{R}_1$, the duality principle implies that $R_{1ppj} = 0$ for all $j, p \neq 1$. Because of (18), this means that $Z_{pp}^{1j} = 0$, for all p and $j \neq 1$.

Conversely, let us assume that $Z_{pp}^{1j} = 0$ for all p and $j \neq 1$, i.e., $R_{1ppj} = 0$. We should show implication (14) for all mutually orthogonal X and $Y \in S_pM$. We can suppose that $X = E_1$ and $Y = E_2$, and Y is an eigenvector of $\mathcal{R}_X = \mathcal{R}_1$, and then, because of the diagonalizability of $\mathcal{R}_1$, we can find an orthonormal frame $(E_1, E_2, \ldots, E_n)$ of T_pM in which $\mathcal{R}_1$ is represented by a diagonal matrix. Then (14) is reduced to (16) for $j = 2$. ◇

Now, we can combine the previous theorem and Theorem 2.2 to obtain some properties of diagonalizable pseudo-Riemannian manifolds.

Corollary 3.4. *If (M, g) be a diagonalizable pseudo-Riemannian Osserman manifold with all different eigenvalues, then the duality principle holds in M.*

Proof. Since M is a diagonalizable manifold and since it has all different eigenvalues, then for any eigenvalue a the set Λ_a has just one element. Formula (12) give us $Z_{pp}^{1j} = 0$ for all p, and Theorem 3.3 implies the statement. ◇

Let us now consider formula (13) from Theorem 2.2, i.e.,

$$\sum_{p,q\in\Lambda_a} Z_{pq}^{1j} Z_{qp}^{1j} = 0.$$

Because $Z_{qp}^{1j} = \varepsilon_q(R_{p1jq} + R_{pj1q}) = \varepsilon_q\varepsilon_p\varepsilon_p(R_{qj1p} + R_{q1jp}) = \varepsilon_p\varepsilon_q Z_{pq}^{1j}$, the previous formula takes the form

$$\sum_{p,q\in\Lambda_a} \varepsilon_p\varepsilon_q (Z_{pq}^{1j})^2 = 0, \tag{19}$$

and we can formulate the following corollary.

Corollary 3.5. *Let (M, g) be a diagonalizable pseudo-Riemannian Osserman manifold such that, for every $X \in S_pM$, a null eigenvector of $\mathcal{R}_X$ does not exist, then the duality principle holds in M. In particular, if M is a Riemannian manifold, then the duality principle holds.*

Proof. Let us choose a basis $(X = E_1, E_2, \ldots, E_n)$ of T_pM in which $\mathcal{R}_X$ has a diagonal matrix. The non-existence of a null eigenvector of $\mathcal{R}_X$ means that all eigenspaces of $\mathcal{R}_X$ contain only spacelike or only timelike vectors. Λ_a generates an eigenspace of $\mathcal{R}_X$ for the eigenvalue a, and we have $\varepsilon_p = \varepsilon_q$ for all $p, q \in \Lambda_a$. Then formula (19) becomes

$$\sum_{p,q\in\Lambda_a} (Z_{pq}^{1j})^2 = 0.$$

From the formula above, it follows that $Z_{pq}^{1j} = 0$ for all $p, q \in \Lambda_a$, which holds for all eigenvalues a of $\mathcal{R}_X$. In particular, $Z_{pp}^{1j} = 0$ for all $p \in \Lambda_a$, and the application of Theorem 3.3 ends this proof. ◇

Remark 3. Similar considerations for non-diagonalizable pseudo-Riemannian Osserman manifolds essentially imply (by the general theory of symmetric endomorphisms in pseudo-unitarian spaces) the study of the duality principle for null vectors. Such manifolds can occur only in neutral signatures (p, p), and its Jordan normal form could be arbitrarily complicated; for more details, see [10].

4. Four-dimensional manifolds

It is well known that an algebraic curvature tensor in dimension four is Osserman if and only if it is Jordan Osserman, but there exist 4-manifolds which are globally Osserman but not globally Jordan Osserman; see Remark 4 and, for more details, [9]. Also, $n = 4$ is the smallest dimension in which there exist timelike (spacelike) Jordan Osserman manifolds with non-diagonalizable Jacobi operator. Algebraic curvature tensors of Osserman (2, 2) manifolds are described in the paper [3]. The geometry of such manifolds is very rich and is studied by many authors; see [9,10,4,1,5,8], etc.

It is known that there are four types of Osserman (2, 2) algebraic curvature tensors. Depending on the roots of the minimal polynomial of its restricted Jacobi operator, they are: (Ia) — diagonalizable case; (Ib) — complex case; (II) — case with a double root; and (III) — case with a triple root. In this section we will use notation introduced in [1,3, 4]. So, we have the following,

Theorem 4.1. *The duality principle holds for every four-dimensional Osserman manifold.*

Proof. If (M, g) is a four-dimensional Osserman manifold, then in the signatures (0, 4) or (1, 3) (the Riemannian (see [20]) and Lorentzian (see [2]) cases) duality holds. So, let us suppose that (M, g) is an Osserman manifold of the signature (2, 2) and we use the classification of algebraic curvature tensors given in [3]. Let us mention here that in case (III) the restricted Jacobi operator does not have any non-null eigenvector (this follows from the general theory of symmetric endomorphisms in pseudo-unitarian spaces (see, [3], Theorem 3.1)) and in this case we have nothing to prove.

(Ia) *M is diagonalizable.* Here, we distinguish cases depending on the number of different eigenvalues of the restricted Jacobi operator. If all three eigenvalues are different (then by Corollary 3.4) or if all three eigenvalues are equal (then M is a space form), duality holds. Because of this, let us suppose that the restricted Jacobi operator $\mathcal{R}'_X$ has exactly two different eigenvalues $\varepsilon_X\lambda$ and $\varepsilon_X\mu$, such that $\varepsilon_X\lambda$ is of multiplicity two. Then μ is an eigenvalue of multiplicity 1, and duality holds (an obvious consequence of Theorem 3.3). So, it remains to prove duality for the eigenvalue λ. Since M is diagonalizable, then for any unit vector $X \in T_pM$ we have the following orthogonal decomposition:

$$T_pM = \langle X\rangle \oplus \mathrm{Ker}(\mathcal{R}'_X - \varepsilon_X\lambda\mathrm{Id}) \oplus \mathrm{Ker}(\mathcal{R}'_X - \varepsilon_X\mu\mathrm{Id}).$$

Let us suppose that $\mathcal{R}_X(Y) = \varepsilon_X\lambda Y$, and consider a unit vector $Y \in \mathrm{Ker}(\mathcal{R}'_X - \varepsilon_X\lambda\,\mathrm{Id})$. Because of the orthogonal decomposition

$$T_pM = \langle Y\rangle \oplus \mathrm{Ker}(\mathcal{R}'_Y - \varepsilon_Y\lambda\,\mathrm{Id}) \oplus \mathrm{Ker}(\mathcal{R}'_Y - \varepsilon_Y\mu\,\mathrm{Id}),$$

and $g(X, Y) = 0$, we have $X = \alpha L + \beta M$, where

$$L \in \mathrm{Ker}(\mathcal{R}'_Y - \varepsilon_Y\lambda\,\mathrm{Id}), \qquad M \in \mathrm{Ker}(\mathcal{R}'_Y - \varepsilon_Y\mu\,\mathrm{Id}), \qquad g(L, M) = 0.$$

Now, we have

$$\begin{aligned}
&\mathcal{R}_Y X = \mathcal{R}_Y(\alpha L + \beta M) = \alpha\mathcal{R}_Y(L) + \beta\mathcal{R}_Y(M) = \alpha\varepsilon_Y\lambda L + \beta\varepsilon_Y\mu M\\
&g(\mathcal{R}_Y X, X) = g(\alpha\varepsilon_Y\lambda L + \beta\varepsilon_Y\mu M, \alpha L + \beta M) = \alpha^2\varepsilon_Y\varepsilon_L\lambda + \beta^2\varepsilon_Y\varepsilon_M\mu
\end{aligned}$$

and since

$$g(\mathcal{R}_Y X, X) = R(X, Y, Y, X) = g(\mathcal{R}_X Y, Y) = g(\varepsilon_X\lambda Y, Y) = \varepsilon_X\varepsilon_Y\lambda$$

it holds that

$$\varepsilon_X\varepsilon_Y\lambda = \alpha^2\varepsilon_Y\varepsilon_L\lambda + \beta^2\varepsilon_Y\varepsilon_M\mu.$$

After substitution of $\varepsilon_X = \alpha^2\varepsilon_L + \beta^2\varepsilon_M$ in the previous equation, one obtains $\beta^2\varepsilon_M\varepsilon_Y\lambda = \beta^2\varepsilon_M\varepsilon_Y\mu$. Space $\mathrm{Ker}(\mathcal{R}'_V - \varepsilon_V\mu\mathrm{Id})$ has dimension 1 and since there are no null vectors we have $\varepsilon_M \neq 0$, then $\lambda \neq \mu$ and $\varepsilon_Y \neq 0$ imply that $\beta = 0$. This means that $X \in \mathrm{Ker}(\mathcal{R}'_Y - \varepsilon_Y\lambda\mathrm{Id})$ and the proof in the diagonalizable case is over.

Cases (Ib) *and* (II). In this case we have the existence of a pseudo-orthonormal basis (see [3]) (E_1, E_2, E_3, E_4) such that $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$ and the Jacobi operator $\mathcal{R}_1 := \mathcal{R}_{E_1}$ with respect to this basis has the following matrix:

$$\mathcal{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a & \gamma & 0 \\ 0 & -\gamma & -b & 0 \\ 0 & 0 & 0 & -c \end{pmatrix}. \tag{20}$$

The use of 1-stein and 2-stein conditions gives (see [3] for more details) all non-vanishing generic components of the curvature tensor

$$\begin{aligned} &R_{1221} = R_{3443} = a, \qquad R_{1331} = R_{2442} = -b, \qquad R_{1441} = R_{2332} = -c, \\ &-R_{2113} = -R_{2443} = R_{1224} = R_{1334} = \gamma, \\ &R_{1234} = x = \frac{-2a+b+c}{3}, \qquad R_{1423} = y = \frac{a+b-2c}{3}, \qquad R_{1342} = -x-y. \end{aligned} \tag{21}$$

In the case (Ib) we have $a = b$, and only a non-trivial (different from E_1) eigenvector of $\mathcal{R}_1$ is E_4. Then we have $\mathcal{R}_1(E_4) = -c\,E_4$, and since $R_{1442} = R_{1443} = R_{1444} = 0$ we conclude that $\mathcal{R}_4(E_1)$ is orthogonal to the subspace $\langle E_2, E_3, E_4\rangle$ and, consequently, $\mathcal{R}_4(E_1) = cE_1$, i.e., duality holds.

Case (II). In this case we have $a = t - \gamma$, $b = t + \gamma$, for some t. Then the eigenvectors of $\mathcal{R}_1$ are $E_2 - E_3$ and E_4.

If a two-dimensional block and a one-dimensional Jordan block of $\mathcal{R}_1$ have different eigenvalues ($t \neq c$) then duality holds, since the eigenvector $E_2 - E_3$ is null (we have nothing to prove) and for the spacelike eigenvector E_4 we obtain (as in the case (Ib)) $\mathcal{R}_4(E_1) = cE_1$.

If a two-dimensional block and a one-dimensional Jordan block of $\mathcal{R}_1$ have the same eigenvalue $-c$ ($t = c$), then from (21) it follows that $y = R_{1432} = 0$. Now, the eigenvectors of $\mathcal{R}_1$ have the following form: $Z = \alpha E_2 - \alpha E_3 + \beta E_4$. Then we have

$$\begin{aligned} \mathcal{R}_1(Z) = \mathcal{R}_1(\alpha E_2 - \alpha E_3 + \beta E_4) &= \alpha(-aE_2 - \gamma E_3 - \gamma E_2 + bE_3) - \beta c E_4 \\ &= -c(\alpha E_2 - \alpha E_3 + \beta E_4) = -cZ = \varepsilon_{E_1} cZ. \end{aligned}$$

Let us find $\mathcal{R}_Z(E_1)$, or equivalently $g(\mathcal{R}_Z(E_1), E_j)$, for $j = 2, 3, 4$:

$$\begin{aligned} T_j &= g(\mathcal{R}_Z(E_1), E_j) \\ &= R(E_1, \alpha E_2 - \alpha E_3 + \beta E_4, \alpha E_2 - \alpha E_3 + \beta E_4, E_j). \end{aligned} \tag{22}$$

Using (21) one can easily find $T_j = 0$ for $j = 2, 3, 4$, and consequently $\mathcal{R}_Z(E_1)$ is orthogonal to the subspace $\langle E_2, E_3, E_4\rangle$ and hence

$$\mathcal{R}_Z(E_1) = c\beta^2 Z = \varepsilon_Z c E_1.$$

If the vector Z is a unit spacelike vector, then $\beta^2 = 1$ and duality in this case also holds. ◇

Remark 4. *Duality and null vectors.* Let us consider the following Walker metric on $\mathbb{R}^4$ which is given in [9], p. 64–66:

$$g = \begin{pmatrix} x_3 f_1 & a & 1 & 0 \\ a & x_4 f_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{23}$$

where $f_1 = f_1(x_1, x_2)$ and $f_2 = f_2(x_1, x_2)$ are smooth real functions and where a is constant. For $\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0$, those metrics define a family of (2, 2) Osserman manifolds.[1] Since pointwise Osserman manifolds are null pointwise

[1] In this family there exist a lot of interesting examples although all manifolds have the same characteristic polynomial λ^4. For example, there exist examples which are locally non-symmetric spaces, which are pointwise Osserman and which are not globally Osserman, etc.

Osserman manifolds (see [10]), the manifolds from the considered class are also null Osserman. Moreover, they satisfy the duality principle for non-null vectors by Theorem 4.1, but they do not satisfy the duality principle for null vectors. For example, the vector E_3 is null and its Jacobi operator $\mathcal{R}_{E_3}$ vanishes, but $\mathcal{R}_{E_1}(E_3) = -\frac{1}{2}\frac{\partial f_1}{\partial x_2} E_4$ (see, p. 66 of [9]); this means that the duality principle is not satisfied for null vector E_3 if $\frac{\partial f_1}{\partial x_2} \neq 0$.

The above example shows that the extension of the duality principle to the null vectors needs additional analysis. The forthcoming paper will deal with a natural extension of the duality principle to null vectors.

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